

Compressed Sensing – the Flowchart

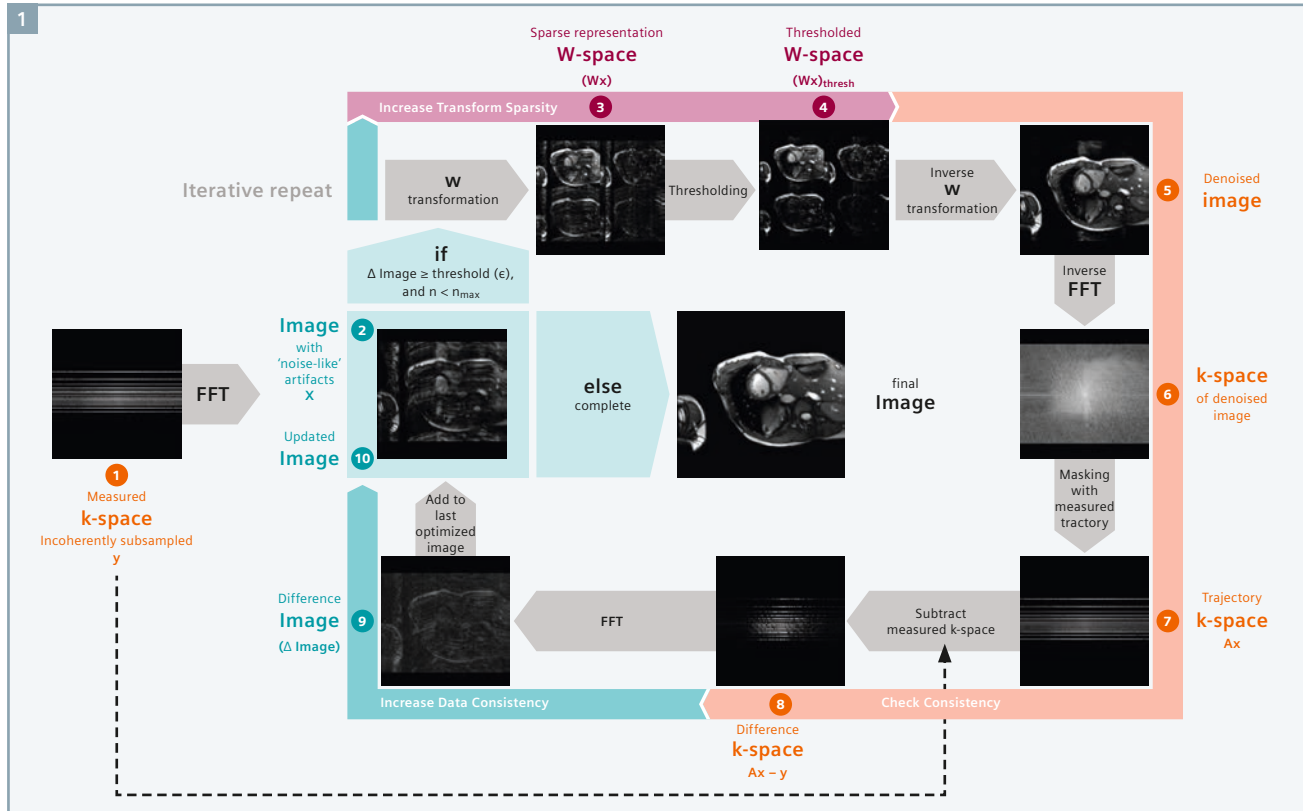
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The three key components of Compressed Sensing (CS)¹ are

1. Incoherent ('random') sub-sampling
2. Transform sparsity
3. Non-linear iterative reconstruction

The Compressed Sensing Flowchart provides a step-by-step visualization of the Compressed Sensing measurement and reconstruction process, explaining where and why these three key components are involved.



1 Compressed Sensing Flowchart. Some 'typical' images of the iterative reconstruction are depicted for visualization purposes.

Abbreviations and terminology

k-space ' y '	Measured data (echoes), sorted in an $m \times n$ matrix; in the formulas denoted as ' y '	A	The transformation A consists of two steps, an inverse FFT and a 'trajectory masking', i.e. only depicting those pixels in k-space that were measured
Image ' x '	Clinical image; in the formulas denoted as ' x '	λ	Weighting factor for the tradeoff between data consistency and transform sparsity
W-space	Different mathematical depiction of the image Examples: Wavelet, Total Variation	$\ ...\ _1$	L1 norm: Sum of all absolute values (here: pixel intensities in W-space)
FFT	(Fast) Fourier transformation; transforms k-space into image space	$\ ...\ _2$	L2 norm, 'Euclidean norm': Square root of sum of squares
FFT ⁻¹	Inverse FFT, transforms image space into k-space		
W	W transformation, transforms image space into W-space		
W ⁻¹	Inverse W Transformation, transforms W-space into image space		

¹ 510(k) pending. Compressed Sensing Cardiac Cine is not commercially available. Future availability cannot be guaranteed.

The Compressed Sensing optimization formula

The following formula describes the optimization process of CS:

$$\min \|Ax - y\|_2^2 + \lambda \|Wx\|_1$$

This means that we are searching for an image x such that the term above is minimized.

The first term describes the 'data consistency'. It minimizes the least-squares difference (L2 norm, $\|\cdot\|_2^2$) between the estimated image x , and the acquired k-space data y . The smaller this (difference) term becomes, the better the consistency.

The second term describes the 'transform sparsity'. It is the L1 norm of the image transformed into a sparse representation (W -space). In this term, the sum of the absolute values of the pixels in the transform domain, denoted by the L1 norm ($\|\cdot\|_1$), is minimized. The smaller this L1 norm, the higher the sparsity. λ is an empirical (application-dependent) weighting factor for balancing data consistency vs. sparsity.

Hence, the optimization procedure minimizing this equation seeks to find a solution that fulfills both criteria, data consistency and transform sparsity. This is done in an iterative process.

1. Measured k-space

The first picture is the measured k-space. There are two differences compared to a 'conventional' scan:

1. Strong sub-sampling, i.e. significantly fewer echoes than in a conventional scan. This results in significantly higher scan speed. But since the Nyquist-Shannon sampling theorem [1] is violated, this will result in strong aliasing artifacts.
2. Incoherent ('random') sampling. This is necessary to create a noise-like appearance of the aliasing artifacts (resulting from the sub-sampling). The reason is that these noise-like artifacts (as opposed to structured arti-

facts) can then in a later step of the algorithm be removed with a thresholding procedure in W -space (see step 4). The random character of the sampling is essential for the success of the CS reconstruction – "Randomness is too important to be left to chance" [2].

The incoherent sampling in CS is different from the (typically) coherent sub-sampling that is used in Parallel Acquisition Techniques, where the (structured) aliasing is removed by means of the knowledge of the coil sensitivity profiles.

The k-space in the formula is denoted as y .

2. Image

As a first step, the (incoherently) subsampled k-space is Fourier transformed into an image.

This image suffers from strong sub-sampling artifacts; but these aliasing artifacts are 'smeared' over the image, due to the incoherence of the sampling in step 1. The aliasing artifacts appear 'noise like'. The better the incoherence of the scan, the more 'homogeneous' the noise-like aliasing artifacts will appear, and the better the CS reconstruction will work.

This image serves as a starting point for the iterative optimization. Since we only covered a small part of k-space in step 1, we do not have the complete information about the image. This image x generated by a straightforward FFT is only one possible solution that is consistent with the measured data – but this solution suffers from strong artifacts. The following iterative process serves to find a better (artifact-free) solution that is also consistent with the measured data y .

The image in the formula is denoted as x .

3. W-space

The image is now transformed into a sparse representation (W -space). This is a different 'basis', i.e. a different mathematical

depiction of the image. The goal of this transformation is to locally separate the 'wanted signal' from the noise (artifacts). W -space is a better-suited depiction of the image as the sparsity in W -space is higher. This means that the image 'informational content' is concentrated in few pixels in W -space, while most pixels have only a very low signal.

There are various different transformations that can be beneficial for this purpose. This depends on the application. A Wavelet transformation is a very common choice for MR imaging.

By the way, a Wavelet transformation is utilized in image compression with the JPEG 2000 format [3].

4. Thresholded W-space

After the W transformation (e.g. Wavelet transformation), the 'wanted signal' is now to a high degree separated from the noise (artifacts). This allows removing the noise by a thresholding procedure:

- Set all pixels with a value $<$ threshold to zero;
- Subtract the threshold from all other pixel values.

This procedure is called 'soft thresholding'. It has been shown that 'soft thresholding' is beneficial for the solution [4]. The threshold is predefined, optimized for the application.

Since many pixels in W -space now have a value of 0, we have fewer non-zero pixels (coefficients). The 'transform sparsity', i.e. the sparsity of the image in W -space, is increased.

5. Denoised image

We now transform the W -space representation back into image space with the inverse W transformation (W^{-1}).

By the thresholding procedure in W -space in the last step, we have now an image with less noise. This corresponds to a suppression of the noise-like aliasing artifacts, due to the incoherent sampling we applied in step 1.

However, with this denoising we have also ‘fiddled’ with the image content. The image has less noise (less artifact power) now, but does not exactly reflect the measurement anymore.

In the next steps 6-8, we will therefore check the image consistency (i.e. how well the denoised image still represents the measurement data).

6. k-space of denoised image

In order to compare the denoised image from step 5 with the measured k-space from step 1, we first apply an inverse Fourier Transformation (FFT⁻¹) to transform the image back to k-space.

Since the image in step 6 was modified, its k-space now consists of all spatial frequencies. This means that we have a ‘complete k-space’ (all pixels in k-space have non-zero values), as opposed to the measured (subsampling) k-space in step 1 which had only few non-zero values.

7. Trajectory k-space

Now follows a masking process. Remember that we measured only a small part of k-space in step 1. The k-space from step 6, on the other hand, is ‘complete’. To compare the two, we filter the k-space (step 6) by only depicting the points (the ‘measurement trajectory’) of k-space (step 6) that were also measured in k-space (step 1). The rest of k-space (step 6) is set to zero.

The two steps inverse FFT (step 6) and masking (step 7) together are called the ‘A matrix’ in the formula. The resulting k-space \mathbf{Ax} can then be directly compared to the measured k-space \mathbf{y} .

8. Difference k-space

We now create the ‘Difference k-space’ by subtracting the k-space \mathbf{Ax} from step 7 from the measured k-space \mathbf{y} from step 1.

The difference ($\mathbf{Ax} - \mathbf{y}$) corresponds to the error (non-consistency) that the thresholding from step 4 has created compared to the measured

k-space \mathbf{y} . The difference is a ‘correction’ k-space, so to speak.

9. Difference image

A simple Fourier transformation converts the ‘Difference k-space’ into a ‘Difference image’.

This is used as a correction for the update of the image that we want to optimize.

10. Updated image

The image \mathbf{x} from step 2 is now updated by adding the correction (difference) image from step 9.

This updated image now has less noise-like artifacts (corresponding to a higher sparsity in W-space) than the image had before the update, due to steps 3 and 4.

At the same time, it was made consistent with the measured k-space from step 1 by means of the correction from steps 8 to 9.

3-10. Iterative reconstruction

Steps 3 to 10 are now repeated. Each iteration will increase the sparsity (in W-space), corresponding to diminishing the aliasing artifacts in image space. At the same time, the consistency of the reconstruction with the measured k-space is taken care of.

This is an alternating optimization of data consistency and transform sparsity, i.e. the two terms in the formula. The factor λ in the formula is a weighting that defines the trade-off between data consistency and sparsity, it is pre-defined and optimized for the application.

As a metaphor:

One can think of scales rocking to and fro (between the optimization of the first term and the optimization of the second term in the formula).

At the same time, the ‘center of gravity’ (the sum of both terms) is moving down (minimizing both terms in the formula simultaneously). The factor λ in the formula can be thought of as a ‘lever’, i.e. a shorter (small λ) or longer (large λ) bar of the scales for

the sparsity term as compared to the data consistency term.

The iteration of steps 3 to 10 is repeated until:

- either the least-squares difference of the data consistency term, i.e., the Difference Image (step 9), is smaller than a predefined threshold ϵ ,

$$\| \mathbf{Ax} - \mathbf{y} \|_2^2 < \epsilon$$

- or a predefined number of iterations N_{\max} is reached.

In the end, we will have an image that is consistent with the measured data, but is denoised (i.e. the noise-like aliasing artifacts have been removed) due to the maximization of the transform sparsity. The final image will (very closely) look as if we had measured k-space completely – but at a much shorter scan time.

Optimization of Compressed Sensing

We have seen that there are many degrees of freedom for the optimization of the Compressed Sensing results. All these need to be taken care of in the application development.

The possible CS acceleration factor depends on the transform sparsity of the dataset. If the acceleration factor is chosen too high (“not justified by the sparsity”), the reconstruction will not yield acceptable results. The possible acceleration factor depends on the application.

The weighting factor λ for the balance between transform sparsity vs. data consistency (as described before) and the threshold for the sparsification in W-space (the soft thresholding in the W-space as discussed earlier) are related to each other in order to achieve optimal results. Also these depend on the application.

All the explanations above are a simplification. There are more ‘intricacies’ that were not discussed for the sake of simplicity, e.g. how the signals from multiple RF channels are handled, the combination of CS with PAT, higher dimensionality, etc.

The CS Flowchart only shows a depiction of a 2D scan (for the sake of simplicity), but Compressed Sensing can also (and better) be applied for multi-dimensional scans with other k-space trajectories. There are two important aspects for the improvement of the CS performance that define which applications benefit most from CS. These are the “Increase of Sparsity” and the “Increase of Randomness”.

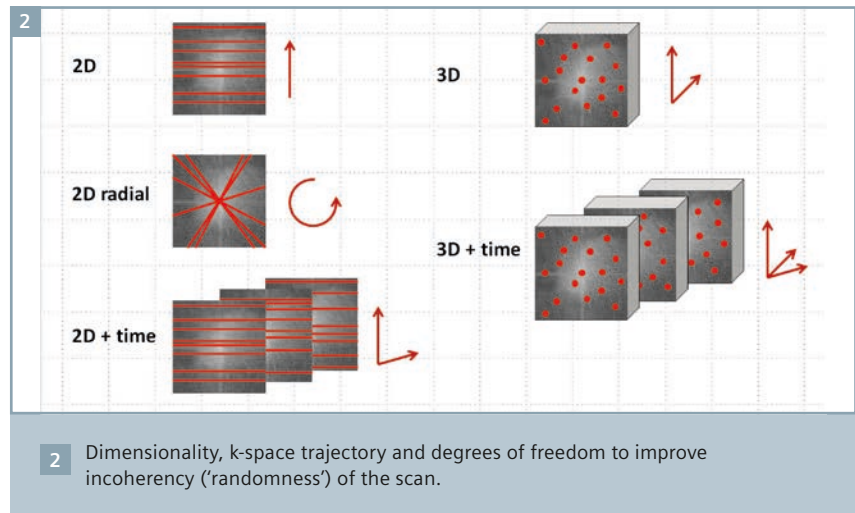
Increase of Sparsity

An “Increase of Sparsity” will enable higher acceleration factors. If the (transform) sparsity of an MR dataset is low, no significant acceleration will be possible, i.e. it will not be possible to remove the noise-like artifacts while preserving an accurate anatomical depiction. For increasing the possible acceleration factor of the scan, a high ‘dimensionality’ of the MR scan helps.

A static 2D scan (like a conventional T2-weighted TSE scan) will typically not have a high sparsity (also not in its Wavelet representation). The achievable acceleration with CS for standard static 2D imaging will therefore not be very high.

But it is very different with e.g. Compressed Sensing Cardiac Cine (2D + time). Along the time dimension, the sparsity is quite high. In CS Cardiac Cine, only little changes are expected between sub-sequent time-frames due to the high temporal resolution and the static anatomy surrounding the heart. This increases the transform sparsity along the time dimension, and therefore high accelerations factors (such as 10) can be achieved in CS Cardiac Cine.

Other examples where the dataset has a high dimensionality and therefore higher sparsity with the potential for



higher acceleration factors include, for example, 3D scans, dynamic ‘4D’ scans (3D + time) and also diffusion imaging with multiple b-values and or multiple diffusion directions.

Increase of Randomness

Also improving the ‘randomness’ of the scan will help the performance and the possible acceleration factors of CS.

Figure 2 shows the number of degrees of freedom for different dimensionalities and different k-space trajectories, in order to improve the ‘randomness’ of the scan.

With static 2D imaging, there is only one degree of freedom, the spacing of the k-space line in phase-encoding direction. The potential for CS acceleration is therefore limited.

With 2D radial imaging, the angle between the different k-space lines can be freely chosen. Each line crosses the center of k-space. There are a couple of advantages for radial imaging vs. cartesian imaging; how-

ever, there are also a couple of disadvantages, such as potential aliasing in both directions and longer reconstruction times. The pros and cons of cartesian vs. radial sampling need to be weighed against each other to find the optimal solution, depending on the application.

2D + time (as in Compressed Sensing Cardiac Cine) also has high dimensionality. There are two degrees of freedom, along the phase-encoding direction and along time. The spatial phase-encoding steps can be chosen differently for each cardiac phase.

Static 3D has two degrees of freedom, since there are two phase-encoding directions. This allows a better ‘randomness’ than static 2D.

Dynamic 3D has an additional degree of freedom, along time. This high dimensionality of course holds great promise for future developments, already available as WIP packages from Siemens.

Overall, there is a wide potential for future Compressed Sensing applications with high acceleration factors and high clinical relevance.

References

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